NEVANLINNA FACTORIZATION AND THE BIEBERBACH CONJECTURE

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A theorem of Arne Beurling [1] determines the invariant subspaces of continuous transformations of a Hilbert space into itself when the factorization theory of functions which are analytic and bounded by one in the unit disk can be applied in a canonical model of the transformation. A determination is now made of the invariant subspaces of continuous transformations of a Hilbert space into itself when the Nevanlinna factorization theory of functions which are analytic and of bounded type in the unit disk can be applied in the canonical model of the transformation. A continuous transformation of a Hilbert space into itself need not have a nontrivial proper closed invariant subspace when the Nevanlinna factorization theory does not apply in the canonical model of the transformation. An estimation theory for functions which are analytic and injective in the unit disk is obtained which generalizes the proof of the Bieberbach conjecture [5].

The Hilbert space of square summable power series is fundamental to applications of the factorization theory of functions which are analytic in the unit disk. The space is the Hilbert space $\mathcal{C}(z)$ of power series

$$f(z) = \sum a_n z^n$$

with complex coefficients for which the sum

$$\langle f(z), f(z) \rangle_{\mathcal{C}(z)} = \sum a_n^- a_n$$

is finite. Summation is over the nonnegative integers n.

A square summable power series $f(z)$ converges in the unit disk and represents a function $f(w)$ of w in the unit disk whose value at w is a scalar product

$$f(w) = \langle f(z), (1 - w^- z)^{-1} \rangle_{\mathcal{C}(z)}$$

with an element

$$(1 - w^- z)^{-1} = \sum (w^n)^- z^n$$

of the Hilbert space. Since the power series is uniquely determined by the function, the power series is frequently identified with the function which it represents. The represented

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function is continuous in the unit disk. It is also differentiable at w when w is the unit disk. The difference quotient

$$\frac{f(z) - f(w)}{z - w}$$

is represented by a square summable power series.

A fundamental theorem of analytic function theory states that a function which is differentiable in the unit disk is represented by a power series. If a function $W(z)$ of z in the unit disk is differentiable and bounded by one, then $W(z)$ is represented by a square summable power series. Proofs of the representation theorem relate geometric properties of functions to their analytic equivalents.

The maximum principle states that a differentiable function $f(z)$ of z in the unit disk, which has a continuous extension to the closure of the unit disk and which is bounded by one on the unit circle is bounded by one in the disk. A contradiction results from the assumption that such a function has values which lie outside of the closure of the unit disk.

Since the function maps the closure of the unit disk onto a compact subset of the complex plane, the complex complement of the set of values is a nonempty open set whose boundary is not contained in the closure of the unit disk. Elements of the unit disk exist which are mapped into the part of the boundary which lies outside of the closed disk. The derivative is easily seen to be zero at such elements of the disk. Such elements a and b of the unit disk are considered equivalent if no disjoint open subsets A and B of the unit disk exist such that a belongs to A, such that b belongs to B, and such that the complement in the disk of the union of A and B is mapped into the closure of the disk. An equivalence relation has been defined on such elements of the disk. Equivalent elements can be reached from each other by a chain in the equivalence class. Since the derivative vanishes on the chain, the function remains constant on the equivalence class. A contradiction is obtained since the function maps the unit disk onto a compact subset of the complex plane whose boundary is contained in the closure of the disk.

An application of the maximum principle is made to a function $W(z)$ of z in the unit disk which is differentiable and bounded by one. If $W(w)$ belongs to the disk for some w in the disk, then the function

$$\frac{W(z) - W(w)}{1 - W(z)W(w)^-}$$

of z in the disk is differentiable and bounded by one. The function $W(z)$ of z maps the unit disk into itself if it is not a constant of absolute value one.

These properties of a function $W(z)$ of z in the unit disk, which are differentiable and bounded by one in the disk, are sufficient [8] for the construction of a Hilbert space $\mathcal{H}(W)$ whose elements are differentiable functions in the disk. The space contains the function

$$\frac{1 - W(z)W(w)^-}{1 - zw^-}$$

of z, when w is in the disk, as reproducing kernel function for function values at w. The identity

$$f(w) = \langle f(z), [1 - W(z)W(w)^-]/(1 - zw^-)\rangle_{\mathcal{H}(W)}$$

holds for every element $f(z)$ of the space. The elements of the space are continuous functions in the disk. The difference quotient

$$\frac{f(z) - f(w)}{z - w}$$

belongs to the space as a function of z when w is in the space. The elements of the space are represented by square summable power series. The space $\mathcal{H}(W)$ is contained contractively in $\mathcal{C}(z)$ when an element of the space is identified with its representing power series. Multiplication by $W(z)$ is a contractive transformation of the space $\mathcal{C}(z)$ into itself.

A power series is treated as a Laurent series which has zero coefficients for negative powers of z. The space of square summable Laurent series is the Hilbert space $\operatorname{ext}\mathcal{C}(z)$ of series

$$\sum a_n z^n$$

defined with summation is over all integers n with a finite sum

$$\|f(z)\|^2_{\operatorname{ext}\mathcal{C}(z)} = \sum a_n^- a_n$$

The space $\mathcal{C}(z)$ of square summable power series is contained isometrically in the space $\operatorname{ext}\mathcal{C}(z)$ of square summable Laurent series. An isometric transformation of $\operatorname{ext}\mathcal{C}(z)$ onto itself, which maps $\mathcal{C}(z)$ onto its orthogonal complement, is defined by taking $f(z)$ into $z^{-1}f(z^{-1})$. The transformation is its own inverse.

Multiplication transformations are defined in the space of square summable power series by power series. The conjugate of a power series

$$W(z) = \sum W_n z^n$$

is the power series

$$W^*(z) = \sum W_n^- z^n$$

whose coefficients are complex conjugate numbers. If $f(z)$ is a power series,

$$g(z) = W(z)f(z)$$

is the power series obtained by Cauchy convolution of coefficients. Multiplication by $W(z)$ in $\mathcal{C}(z)$ is the transformation which takes $f(z)$ into $g(z)$ when $f(z)$ and $g(z)$ belongs to $\mathcal{C}(z)$. Multiplication by $W(z)$ in $\mathcal{C}(z)$ is said to be a Toeplitz transformation if it has domain dense in $\mathcal{C}(z)$. If multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$, then the adjoint is a transformation whose domain contains the polynomial elements of $\mathcal{C}(z)$. The adjoint transformation maps a polynomial element $f(z)$ of $\mathcal{C}(z)$ into the polynomial element $g(z)$ of $\mathcal{C}(z)$ such that

$$z^{-1}g(z^{-1}) - W^*(z)z^{-1}f(z^{-1})$$

is a power series. Multiplication by $W(z)$ in $\mathcal{C}(z)$ is then the adjoint of its adjoint restricted to polynomial elements of $\mathcal{C}(z)$.

A Krein space $\mathcal{H}(W)$, whose elements are power series, is constructed from a given power series $W(z)$ when multiplication by $W(z)$ is a densely defined transformation in $\mathcal{C}(z)$. The space contains

$$f(z) - W(z)g(z)$$

whenever $f(z)$ and $g(z)$ are elements of $\mathcal{C}(z)$ such that the adjoint of multiplication by $W(z)$ in $\mathcal{C}(z)$ takes $f(z)$ into $g(z)$ and such that $g(z)$ is in the domain of multiplication by $W(z)$ in $\mathcal{C}(z)$. The identity

$$\langle h(z), f(z) - W(z)g(z)\rangle_{\mathcal{H}(W)} = \langle h(z), f(z)\rangle_{\mathcal{C}(z)}$$

then holds for every element $h(z)$ of the space $\mathcal{H}(W)$ which belongs to $\mathcal{C}(z)$. The series $[f(z)-f(0)]/z$ belongs to the space $\mathcal{H}(W)$ whenever $f(z)$ belongs to the space. The Krein space $\mathcal{H}(W')$ associated with the power series

$$W'(z) = zW(z)$$

is the set of power series $f(z)$ with vector coefficients such that $[f(z) - f(0)]/z$ belongs to the space $\mathcal{H}(W)$. The identify for difference quotients

$$\langle [f(z) - f(0)]/z,\ [f(z) - f(0)]/z\rangle_{\mathcal{H}(W)} = \langle f(z), f(z)\rangle_{\mathcal{H}(W')} - f(0)^{-}f(0)$$

is then satisfied. The resulting properties of the space $\mathcal{H}(W)$ create [4] a canonical coisometric linear system with transfer function $W(z)$. The space $\mathcal{H}(W)$ is the state space of the linear system. The main transformation, which maps the state space into itself, takes $f(z)$ into

$$[f(z) - f(0)]/z.$$

The input transformation, which maps the space of complex numbers into the state space, takes c into

$$[W(z) - W(0)]c/z.$$

The output transformation, which maps the state space into the space of complex numbers, takes $f(z)$ into $f(0)$. The external operator, which maps the space of complex numbers into itself, takes c into

$$W(0)c.$$

A matrix of continuous linear transformations has been constructed which maps the Cartesian product of the state space and the space of complex numbers continuously into itself. The coisometric property of the linear system states that the matrix has an isometric adjoint.

A Krein space is a vector space with scalar product which is the orthogonal sum of a Hilbert space and the anti–space of a Hilbert space. A Krein space is characterized as a vector space with scalar product which is self–dual for a norm topology.

Theorem 1. *A vector space with scalar product is a Krein space if it admits a norm which satisfies the convexity identity*

$$\|(1-t)a + tb\|^2 + t(1-t)\|b-a\|^2 = (1-t)\|a\|^2 + t\|b\|^2$$

for all elements a and b of the space when $0 < t < 1$ and if the linear functionals on the space which are continuous for the metric topology defined by the norm are the linear functionals which are continuous for the weak topology induced by duality of the space with itself.

Proof of Theorem 1. Norms on the space are considered which satisfy the hypotheses of the theorem. The hypotheses imply that the space is complete in the metric topology defined by any such norm. If a norm $\|c\|_+$ is given for elements c of the space, a dual norm $\|c\|_-$ for elements c of the space is defined by the least upper bound

$$\|a\|_- = \sup |\langle a, b\rangle|$$

taken over the elements b of the space such that

$$\|b\|_+ < 1.$$

The least upper bound is finite since every linear functional which is continuous for the weak topology induced by self–duality is assumed continuous for the metric topology. Since every linear functional which is continuous for the metric topology is continuous for the weak topology induced by self–duality, the set of such elements b is a disk for the weak topology induced by self–duality. The set of elements a of the space such that

$$\|a\|_- \leq 1$$

is compact in the weak topology induced by self–duality. The set of elements a of the space such that

$$\|a\|_- < 1$$

is open for the metric topology induced by the plus norm. Since the set is a disk for the weak topology induced by self–duality, the set of elements b of the space such that

$$\|b\|_+ \leq 1$$

is compact in the weak topology induced by self–duality.

The convexity identity

$$\|(1-t)a + tb\|_+^2 + t(1-t)\|b-a\|_+^2 = (1-t)\|a\|_+^2 + t\|b\|_+^2$$

holds by hypothesis for all elements a and b of the space when $0 < t < 1$. It will be shown that the convexity identity

$$\|(1-t)u + tv\|_-^2 + t(1-t)\|v-u\|_-^2 = (1-t)\|u\|_-^2 + t\|v\|_-^2$$

holds for all elements u and v of the space when $0 < t < 1$. Use is made of the convexity identity

$$\langle (1-t)a + tb, (1-t)u + tv\rangle + t(1-t)\langle b - a, v - u\rangle$$
$$= (1-t)\langle a, u\rangle + t\langle b, v\rangle$$

for elements a, b, u, and v of the space when $0 < t < 1$. Since the inequality

$$|(1-t)\langle a, u\rangle + t\langle b, v\rangle|$$
$$\leq \|(1-t)a + tb\|_+ \|(1-t)u + tv\|_- + t(1-t)\|b - a\|_+ \|v - u\|_-$$

holds by the definition of the minus norm, the inequality

$$|(1-t)\langle a, u\rangle + t\langle b, v\rangle|^2 \leq [\|(1-t)a + tb\|_+^2 + t(1-t)\|b - a\|_+^2]$$
$$\times [\|(1-t)u + tv\|_-^2 + t(1-t)\|v - u\|_-^2]$$

is satisfied. The inequality

$$|(1-t)\langle a, u\rangle + t\langle b, v\rangle|^2 \leq [(1-t)\|a\|_+^2 + t\|b\|_+^2]$$
$$\times [\|(1-t)u + tv\|_-^2 + t(1-t)\|v - u\|_-^2]$$

holds by the convexity identity for the plus norm. The inequality is applied for all elements a and b of the space such that the inequalities

$$\|a\|_+ \leq \|u\|_-$$

and

$$\|b\|_+ \leq \|v\|_-$$

are satisfied. The inequality

$$(1-t)\|u\|_-^2 + t\|v\|_-^2 \leq \|(1-t)u + tv\|_-^2 + t(1-t)\|v - u\|_-^2$$

follows by the definition of the minus norm. Equality holds since the reverse inequality is a consequence of the identities

$$(1-t)[(1-t)u + tv] + t[(1-t)u - (1-t)v] = (1-t)u$$

and

$$[(1-t)u + tv] - [(1-t)u - (1-t)v] = v.$$

It has been verified that the minus norm satisfies the hypotheses of the theorem. The dual norm to the minus norm is the plus norm. Another norm which satisfies the hypotheses of the theorem is defined by

$$\|c\|_t^2 = (1-t)\|c\|_+^2 + t\|c\|_-^2$$

when $0 < t < 1$. Since the inequalities

$$|\langle a, b\rangle| \leq \|a\|_+ \|b\|_-$$

and

$$|\langle a, b\rangle| \leq \|a\|_- \|b\|_+$$

hold for all elements a and b of the space, the inequality

$$|\langle a, b\rangle| \leq (1-t)\|a\|_+ \|b\|_- + t\|a\|_- \|b\|_+$$

holds when $0 < t < 1$. The inequality

$$|\langle a, b\rangle| \leq \|a\|_t \|b\|_{1-t}$$

follows for all elements a and b of the space when $0 < t < 1$. The inequality implies that the dual norm of the t norm is dominated by the $1-t$ norm. A norm which dominates its dual norm is obtained when $t = \frac{1}{2}$.

Consider the norms which satisfy the hypotheses of the theorem and which dominate their dual norms. Since a nonempty totally ordered set of such norms has a greatest lower bound, which is again such a norm, a minimal such norm exists by the Zorn lemma. If a minimal norm is chosen as the plus norm, it is equal to the t–norm obtained when $t = \frac{1}{2}$. It follows that a minimal norm is equal to its dual norm.

If a norm satisfies the hypotheses of the theorem and is equal to its dual norm, a related scalar product is introduced on the space which may be different from the given scalar product. Since the given scalar product assumes a subsidiary role in the subsequent argument, it is distinguished by a prime. A new scalar product is defined by the identity

$$4\langle a, b\rangle = \|a+b\|^2 - \|a-b\|^2 + i\|a+ib\|^2 - i\|a-ib\|^2.$$

The symmetry of a scalar product is immediate. Linearity will be verified.

The identity

$$\langle wa, wb\rangle = w^- w\langle a, b\rangle$$

holds for all elements a and b of the space if w is a complex number. The identity

$$\langle ia, b\rangle = i\langle a, b\rangle$$

holds for all elements a and b of the space. The identity

$$\langle ta, b\rangle = t\langle a, b\rangle$$

will be verified for all elements a and b of the space when t is a positive number. It is sufficient to verify the identity

$$\|ta+b\|^2 - \|ta-b\|^2 = t\|a+b\|^2 - t\|a-b\|^2$$

since a similar identity follows with b replaced by ib. The identity holds since

$$\|ta+b\|^2 + t\|a-b\|^2 = t(1+t)\|a\|^2 + (1+t)\|b\|^2$$

and

$$\|ta-b\|^2 + t\|a+b\|^2 = t(1+t)\|a\|^2 + (1+t)\|b\|^2$$

by the convexity identity.

If a, b, and c are elements of the space and if $0 < t < 1$, the identity

$$4\langle (1-t)a + tb, c\rangle = \|(1+t)(a+c) + t(b+c)\|^2$$
$$-\|(1-t)(a-c) + t(b-c)\|^2 + i\|(1-t)(a+ic) + t(b+ic)\|^2$$
$$-i\|(1-t)(a-ic) + t(b-ic)\|^2$$

is satisfied with the right side equal to

$$(1-t)\|a+c\|^2 + t\|b+c\|^2 - (1-t)\|a-c\|^2 - t\|b-c\|^2$$
$$+i(1-t)\|a+ic\|^2 + it\|b+ic\|^2 - i(1-t)\|a-ic\|^2 - it\|b-ic\|^2$$
$$= 4(1-t)\langle a, c\rangle + 4t\langle b, c\rangle.$$

The identity

$$\langle (1-t)a + tb, c\rangle = (1-t)\langle a, c\rangle + t\langle b, c\rangle$$

follows.

Linearity of a scalar product is now easily verified. Scalar self–products are nonnegative since the identity

$$\langle c, c\rangle = \|c\|^2$$

holds a for every element c of the space. A Hilbert space is obtained whose norm is the minimal norm. Since the inequality

$$|\langle a, b\rangle'| \leq \|a\|\|b\|$$

holds for all elements a and b of the space, a contractive transformation J of the Hilbert space into itself exists such that the identity

$$\langle a, b\rangle' = \langle Ja, b\rangle$$

holds for all elements a and b of the space. The symmetry of the given scalar product implies that the transformation J is self–adjoint. Since the Hilbert space norm is self–dual with respect to the given scalar product, the transformation J is also isometric with respect to the Hilbert space scalar product. The space is the orthogonal sum of the space of eigenvectors of J for the eigenvalue one and the space of eigenvectors of J for the eigenvalue minus one. These spaces are also orthogonal with respect to the given scalar product. They are the required Hilbert space and anti–space of a Hilbert space for the orthogonal decomposition of the vector space with scalar product to form a Krein space.

This completes the proof of the theorem.

The orthogonal decomposition of a Krein space is not unique since equivalent norms can be used. The dimension of the anti–space of a Hilbert space in the decomposition is however an invariant called the Pontryagin index of the Krein space. Krein spaces are a natural setting for a complementation theory which was discovered in Hilbert spaces [3].

A generalization of the concept of orthogonal complement applies when a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$. The contractive property of the inclusion means that the inequality

$$\langle a, a\rangle_{\mathcal{H}} \leq \langle a, a\rangle_{\mathcal{P}}$$

holds for every element a of $\mathcal{P}$. Continuity of the inclusion means that an adjoint transformation of $\mathcal{H}$ into $\mathcal{P}$ exists. A self–adjoint transformation $\mathcal{P}$ of $\mathcal{H}$ into $\mathcal{H}$ is obtained on composing the inclusion with the adjoint. The inequality

$$\langle Pc, Pc\rangle_{\mathcal{H}} \leq \langle Pc, Pc\rangle_{\mathcal{P}}$$

for elements c of $\mathcal{H}$ implies the inequality

$$\langle P^2c, c\rangle_{\mathcal{H}} \leq \langle Pc, c\rangle_{\mathcal{H}}$$

for elements c of $\mathcal{H}$, which is restated as an inequality

$$P^2 \leq P$$

for self–adjoint transformations in $\mathcal{H}$.

The properties of adjoint transformations are used in the construction of a complementary space $\mathcal{Q}$ to $\mathcal{P}$ in $\mathcal{H}$.

Theorem 2. *If a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$, then a unique Krein space $\mathcal{Q}$ exists, which is contained continuously and contractively in $\mathcal{H}$, such that the inequality*

$$\langle c, c\rangle_{\mathcal{H}} \leq \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

holds whenever $c = a + b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some such decomposition for which equality holds.

Proof. Define $\mathcal{Q}$ to be the set of elements b of $\mathcal{H}$ such that the least upper bound

$$\langle b, b\rangle_{\mathcal{Q}} = \sup[\langle a + b, a + b\rangle_{\mathcal{H}} - \langle a, a\rangle_{\mathcal{P}}]$$

taken over all elements a of $\mathcal{P}$ is finite. It will be shown that $\mathcal{Q}$ is a vector space with scalar product having the desired properties. Since the origin belongs to $\mathcal{P}$, the inequality

$$\langle b, b\rangle_{\mathcal{H}} \leq \langle b, b\rangle_{\mathcal{Q}}$$

holds for every element b of $\mathcal{Q}$. Since the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is contractive, the origin belongs to $\mathcal{Q}$ and has self–product zero. If b belongs to $\mathcal{Q}$ and if w is a complex number, then wb is an element of $\mathcal{Q}$ which satisfies the identity

$$\langle wb, wb\rangle_{\mathcal{Q}} = w^{-}w\langle b, b\rangle_{\mathcal{Q}}.$$

The set $\mathcal{Q}$ is invariant under multiplication by complex numbers. The set $\mathcal{Q}$ is shown to be a vector space by showing that it is closed under convex combinations.

It will be shown that $(1-t)a+tb$ belongs to $\mathcal{Q}$ whenever a and b are elements of $\mathcal{Q}$ and t is a number, $0 < t < 1$. Since an arbitrary pair of elements of $\mathcal{P}$ can be written in the form $(1-t)a+tv$ and $v-u$ for elements u and v of $\mathcal{P}$, the identity

$$\begin{aligned}
&\langle (1-t)a+tb, (1-t)a+tb\rangle_{\mathcal{Q}} + t(1-t)\langle b-a, b-a\rangle_{\mathcal{Q}} \\
&= \sup[\langle (1-t)(a+u)+t(b+v), (1-t)(a+u)+t(b+v)\rangle_{\mathcal{H}} \\
&\quad + t(1-t)\langle (b+v)-(a+u), (b+v)-(a+u)\rangle_{\mathcal{H}} \\
&\quad - \langle (1-t)u+tv, (1-t)u+tv\rangle_{\mathcal{P}} - t(1-t)\langle v-u, v-u\rangle_{\mathcal{P}}]
\end{aligned}$$

holds with the least upper bound taken over all elements u and v of $\mathcal{P}$. By the convexity identity the least upper bound

$$\begin{aligned}
&\langle (1-t)a+tb, (1-t)a+tb\rangle_{\mathcal{Q}} + t(1-t)\langle b-a, b-a\rangle_{\mathcal{Q}} \\
&= \sup[\langle a+u, a+u\rangle_{\mathcal{H}} - \langle u, u\rangle_{\mathcal{P}}] + \sup[\langle b+v, b+v\rangle_{\mathcal{H}} - \langle v, v\rangle_{\mathcal{P}}]
\end{aligned}$$

holds over all elements u and v of $\mathcal{P}$. It follows that the identity

$$\begin{aligned}
&\langle (1-t)a+tb, (1-t)a+tb\rangle_{\mathcal{Q}} + t(1-t)\langle b-a, b-a\rangle_{\mathcal{Q}} \\
&= (1-t)\langle a, a\rangle_{\mathcal{Q}} + t\langle b, b\rangle_{\mathcal{Q}}
\end{aligned}$$

is satisfied.

This completes the verification that $\mathcal{Q}$ is a vector space. It will be shown that a scalar product is defined on the space by the identity

$$4\langle a, b\rangle_{\mathcal{Q}} = \langle a+b, a+b\rangle_{\mathcal{Q}} - \langle a-b, a-b\rangle_{\mathcal{Q}} + i\langle a+ib, a+ib\rangle_{\mathcal{Q}} - i\langle a-ib, a-ib\rangle_{\mathcal{Q}}.$$

Linearity and symmetry of a scalar product are verified as in the characterization of Krein spaces. The nondegeneracy of a scalar product remains to be verified.

Since the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is continuous, a self–adjoint transformation P of $\mathcal{H}$ into itself exists which coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. If c is an element of $\mathcal{H}$ and if a is an element of $\mathcal{P}$, the inequality

$$\langle a-Pc, a-Pc\rangle_{\mathcal{H}} \leq \langle a-Pc, a-Pc\rangle_{\mathcal{P}}$$

implies the inequality

$$\langle (1-P)c, (1-P)c\rangle_{\mathcal{Q}} \le \langle c, c\rangle_{\mathcal{H}} - \langle Pc, Pc\rangle_{\mathcal{P}}.$$

Equality holds since the reverse inequality follows from the definition of the self–product in $\mathcal{Q}$. If b is an element of $\mathcal{Q}$ and if c is an element of $\mathcal{H}$, the inequality

$$\langle b-c, b-c\rangle_{\mathcal{H}} \le \langle Pc, Pc\rangle_{\mathcal{P}} + \langle b-(1-P)c, b-(1-P)c\rangle_{\mathcal{Q}}$$

can be written

$$\langle b, b\rangle_{\mathcal{H}} - \langle b, c\rangle_{\mathcal{H}} - \langle c, b\rangle_{\mathcal{H}} \le \langle b, b\rangle_{\mathcal{Q}} - \langle b, (1-P)c\rangle_{\mathcal{Q}} - \langle (1-P)c, b\rangle_{\mathcal{Q}}.$$

Since b can be replaced by wb for every complex number w, the identity

$$\langle b, c\rangle_{\mathcal{H}} = \langle b, (1-P)c\rangle_{\mathcal{Q}}$$

is satisfied. The nondegeneracy of a scalar product follows in the space $\mathcal{Q}$. The space $\mathcal{Q}$ is contained continuously in the space $\mathcal{H}$ since $1-P$ coincides with the adjoint of the inclusion of $\mathcal{Q}$ in the space $\mathcal{H}$.

The intersection of $\mathcal{P}$ and $\mathcal{Q}$ is considered as a vector space $\mathcal{P} \wedge \mathcal{Q}$ with scalar product

$$\langle a, b\rangle_{\mathcal{P}\wedge\mathcal{Q}} = \langle a, b\rangle_{\mathcal{P}} + \langle a, b\rangle_{\mathcal{Q}}.$$

Linearity and symmetry of a scalar product are immediate, but nondegeneracy requires verification. If c is an element of $\mathcal{H}$,

$$P(1-P)c = (1-P)Pc$$

is an element of $\mathcal{P} \wedge \mathcal{Q}$ which satisfies the identity

$$\langle a, P(1-P)c\rangle_{\mathcal{P}\wedge\mathcal{Q}} = \langle a, c\rangle_{\mathcal{H}}$$

for every element a of $\mathcal{P} \wedge \mathcal{Q}$. Nondegeneracy of a scalar product in $\mathcal{P} \wedge \mathcal{Q}$ follows from nondegeneracy of the scalar product in $\mathcal{H}$. The space $\mathcal{P} \wedge \mathcal{Q}$ is contained continuously in the space $\mathcal{H}$. The self–adjoint transformation $P(1-P)$ in $\mathcal{H}$ coincides with the adjoint of the inclusion of $\mathcal{P} \wedge \mathcal{Q}$ in $\mathcal{H}$. The inequality

$$0 \le \langle c, c\rangle_{\mathcal{P}\wedge\mathcal{Q}}$$

holds for every element c of $\mathcal{P} \wedge \mathcal{Q}$ since the identity

$$0 = c - c$$

with c in $\mathcal{P}$ and $-c$ in $\mathcal{Q}$ implies the inequality

$$0 \le \langle c, c\rangle_{\mathcal{P}} + \langle c, c\rangle_{\mathcal{Q}}.$$

It will be shown that the space $\mathcal{P} \wedge \mathcal{Q}$ is a Hilbert space. The metric topology of the space is the disk topology resulting from duality of the space with itself. Since the inclusion of $P \wedge \mathcal{Q}$ in $\mathcal{P}$ is continuous from the weak topology induced by $\mathcal{P} \wedge \mathcal{Q}$ into the weak topology induced by $\mathcal{P}$, it is continuous from the disk topology induced by $\mathcal{P} \wedge \mathcal{Q}$ into the disk topology induced by $\mathcal{P}$. Since $\mathcal{P}$ is a Krein space, it is complete in its disk topology. A Cauchy sequence of elements c_n of $\mathcal{P} \wedge \mathcal{Q}$ is then a convergent sequence of elements of $\mathcal{P}$. The limit is an element c of $\mathcal{P}$ such that the identity

$$\langle c, a\rangle_{\mathcal{P}} = \lim \langle c_n, a\rangle_{\mathcal{P}}$$

holds for every element a of $\mathcal{P}$ and such that the identity

$$\langle c, c\rangle_{\mathcal{P}} = \lim \langle c_n, c_n\rangle_{\mathcal{P}}$$

is satisfied. Since the inclusion of $\mathcal{P}$ in $\mathcal{H}$ is continuous from the disk topology of $\mathcal{P}$ into the disk topology of $\mathcal{H}$, the identity

$$\langle c, a\rangle_{\mathcal{H}} = \lim \langle c_n, a\rangle_{\mathcal{H}}$$

holds for every element a of $\mathcal{H}$ and the identity

$$\langle c, c\rangle_{\mathcal{H}} = \lim \langle c_n, c_n\rangle_{\mathcal{H}}$$

is satisfied.

If b is an element of $\mathcal{Q}$, the limits

$$\lim \langle c_n, b\rangle_{\mathcal{Q}}$$

and

$$\lim \langle c_n, c_n\rangle_{\mathcal{Q}}$$

exist since the inclusion of $\mathcal{P} \wedge \mathcal{Q}$ in $\mathcal{Q}$ is continuous from the disk topology of $\mathcal{P} \wedge \mathcal{Q}$ into the disk topology of $\mathcal{Q}$. The sequence of elements c_n of $\mathcal{Q}$ is Cauchy in the disk topology of $\mathcal{Q}$. If a is an element of $\mathcal{P}$, the identity

$$\langle a + c, a + c\rangle_{\mathcal{H}} = \lim \langle a + c_n, a + c_n\rangle_{\mathcal{H}}$$

is satisfied. Since the inequality

$$\langle a + c_n, a + c_n\rangle_{\mathcal{H}} - \langle a, a\rangle_{\mathcal{P}} \leq \langle c_n, c_n\rangle_{\mathcal{Q}}$$

holds for every index n, the inequality

$$\langle a + c, a + c\rangle_{\mathcal{H}} - \langle a, a\rangle_{\mathcal{P}} \leq \lim \langle c_n, c_n\rangle_{\mathcal{Q}}$$

is satisfied. It follows that c belongs to $\mathcal{Q}$ and that

$$\langle c, c\rangle_{\mathcal{Q}} \leq \lim \langle c_n, c_n\rangle_{\mathcal{Q}}.$$

Since the inequality

$$\langle c - c_m, c - c_m \rangle_{\mathcal{Q}} \leq \lim \langle c_n - c_m, c_n - c_m \rangle_{\mathcal{Q}}$$

holds for every index m and since the elements c_n of $\mathcal{Q}$ form a Cauchy sequence in the disk topology of $\mathcal{Q}$, the limit of the elements c_n of $\mathcal{Q}$ is equal to c. This completes the proof that $\mathcal{P} \wedge \mathcal{Q}$ is a Hilbert space.

The Cartesian product of $\mathcal{P}$ and $\mathcal{Q}$ is isomorphic to the Cartesian product of $\mathcal{H}$ and $\mathcal{P} \wedge \mathcal{Q}$. If a is an element of $\mathcal{P}$ and if b is an element of $\mathcal{Q}$, a unique element c of $\mathcal{P} \wedge \mathcal{Q}$ exists such that the identity

$$\langle a - c, a - c \rangle_{\mathcal{P}} + \langle b + c, b + c \rangle_{\mathcal{Q}} = \langle a + b, a + b \rangle_{\mathcal{H}} + \langle c, c \rangle_{\mathcal{P} \wedge \mathcal{Q}}$$

is satisfied. Every element of the Cartesian product of $\mathcal{H}$ and $\mathcal{P} \wedge \mathcal{Q}$ is a pair $(a + b, c)$ for elements a of $\mathcal{P}$ and b of $\mathcal{Q}$ for such an element c of $\mathcal{P} \wedge \mathcal{Q}$. Since $\mathcal{H}$ is a Krein space and since $\mathcal{P} \wedge \mathcal{Q}$ is a Hilbert space, the Cartesian product of $\mathcal{P}$ and $\mathcal{Q}$ is a Krein space. Since $\mathcal{P}$ is a Krein space, it follows that $\mathcal{Q}$ is a Krein space.

The existence of a Krein space $\mathcal{Q}$ with the desired properties has now been verified. Uniqueness is proved by showing that a Krein space $\mathcal{Q}'$ with these properties is isometrically equal to the space $\mathcal{Q}$ constructed. Such a space $\mathcal{Q}'$ is contained contractively in the space $\mathcal{Q}$. The self–adjoint transformation $1 - P$ in $\mathcal{H}$ coincides with the adjoint of the inclusion of $\mathcal{Q}'$ in $\mathcal{H}$. The space $\mathcal{P} \wedge \mathcal{Q}'$ is a Hilbert space which is contained contractively in the Hilbert space $\mathcal{P} \wedge \mathcal{Q}$. Since the inclusion is isometric on the range of $P(1 - P)$, which is dense in both spaces, the space $\mathcal{P} \wedge \mathcal{Q}'$ is isometrically equal to the space $\mathcal{P} \wedge \mathcal{Q}$. Since the Cartesian product of $\mathcal{P}$ and $\mathcal{Q}'$ is isomorphic to the Cartesian product of $\mathcal{P}$ and $\mathcal{Q}$, the spaces $\mathcal{Q}$ and $\mathcal{Q}'$ are isometrically equal.

This completes the proof of the theorem.

The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. The space $\mathcal{P}$ is recovered as the complementary space to the space $\mathcal{Q}$ in $\mathcal{H}$. The decomposition of an element c of $\mathcal{H}$ as $c = a + b$ with a an element of $\mathcal{P}$ and b an element of $\mathcal{Q}$ such that equality hold in the inequality

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

is unique. The minimal decomposition results when a is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and b is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

A construction is made of complementary subspaces whose inclusion in the full space have adjoints coinciding with given self–adjoint transformations.

Theorem 3. *If a self–adjoint transformation P of a Krein space into itself satisfies the inequality*

$$P^2 \leq P,$$

then unique Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ exist, which are contained continuously and contractively in $\mathcal{H}$ and which are complementary spaces in $\mathcal{H}$, such that P coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and $1-P$ coincides with the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

Proof of Theorem 3. The proof repeats the construction of a complementary space under a weaker hypothesis. The range of P is considered as a vector space $\mathcal{P}'$ with scalar product determined by the identity

$$\langle Pc, Pc\rangle_{\mathcal{P}'} = \langle Pc, c\rangle_{\mathcal{H}},$$

for every element c of $\mathcal{H}$. The space $\mathcal{P}'$ is contained continuously and contractively in the space $\mathcal{H}$. The transformation P coincides with the adjoint of the inclusion of $\mathcal{P}'$ in $\mathcal{H}$. A Krein space $\mathcal{Q}$, which is contained continuously and contractively in $\mathcal{H}$, is defined as the set of elements b of $\mathcal{H}$ such that the least upper bound

$$\langle b, b\rangle_{\mathcal{Q}} = \sup[\langle a+b, a+b\rangle_{\mathcal{H}} - \langle a, a\rangle_{\mathcal{P}'}]$$

taken over all elements a of $\mathcal{P}'$ is finite. The adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$ coincides with $1-P$. The complementary space to $\mathcal{Q}$ in $\mathcal{H}$ is a Krein space $\mathcal{P}$ which contains the space $\mathcal{P}'$ isometrically and which is contained continuously and contractively in $\mathcal{H}$. The adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ coincides with $1-P$.

This completes the proof of the theorem.

A factorization of continuous and contractive transformations in Krein spaces is an application of complementation theory.

Theorem 4. *The kernel of a continuous and contractive transformation T of a Krein space $\mathcal{P}$ into a Krein space $\mathcal{Q}$ is a Hilbert space which is contained continuously and isometrically in $\mathcal{P}$ and whose orthogonal complement in $\mathcal{P}$ is mapped isometrically onto a Krein space which is contained continuously and contractively in $\mathcal{Q}$.*

Proof of Theorem 4. Since the transformation T of $\mathcal{P}$ into $\mathcal{Q}$ is continuous and contractive, the self–adjoint transformation $P = TT^*$ in $\mathcal{Q}$ satisfies the inequality $P^2 \le P$. A unique Krein space $\mathcal{M}$, which is contained continuously and contractively in $\mathcal{Q}$, exists such that $\mathcal{P}$ coincides with the adjoint of the inclusion of $\mathcal{M}$ in $\mathcal{Q}$. It will be shown that T maps $\mathcal{P}$ contractively into $\mathcal{M}$.

If a is an element of $\mathcal{P}$ and if b is an element of $\mathcal{Q}$, then

$$\langle Ta + (1-P)b, Ta + (1-P)b\rangle_{\mathcal{Q}}$$
$$= \langle T(a-T^*b), T(a-T^*b)\langle_{\mathcal{Q}} + \rangle b, b\rangle_{\mathcal{Q}} + \langle b, T(a-T^*b)\rangle_{\mathcal{Q}} + \langle T(a-T^*b), b\rangle_{\mathcal{Q}}$$

is less than or equal to

$$\langle a-T^*b, a-T^*b\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}} + \langle T^*b, a-T^*b\rangle_{\mathcal{P}} + \langle a-T^*b, T^*b\rangle_{\mathcal{P}}$$
$$= \langle a, a\rangle_{\mathcal{P}} + \langle (1-TT^*)b, b\rangle_{\mathcal{Q}}.$$

Since b is an arbitrary element of $\mathcal{Q}$, Ta is an element of $\mathcal{M}$ which satisfies the inequality

$$\langle Ta, Ta\rangle_{\mathcal{M}} \leq \langle a, a\rangle_{\mathcal{P}}.$$

Equality holds when $a = T^*b$ for an element b of $\mathcal{Q}$ since

$$\langle TT^*b, TT^*b\rangle_{\mathcal{M}} = \langle TT^*b, b\rangle_{\mathcal{Q}} = \langle T^*b, T^*b\rangle_{\mathcal{P}}.$$

Since the transformation of $\mathcal{P}$ into $\mathcal{M}$ is continuous by the closed graph theorem, the adjoint transformation is an isometry. The range of the adjoint transformation is a Krein space which is contained continuously and isometrically in $\mathcal{P}$ and whose orthogonal complement is the kernel of T. Since T is contractive, the kernel of T is a Hilbert space.

This completes the proof of the theorem.

A continuous transformation of a Krein space $\mathcal{P}$ into a Krein space $\mathcal{Q}$ is said to be a partial isometry if its kernel is a Krein space which is contained continuously and isometrically in $\mathcal{P}$ and whose orthogonal complement is mapped isometrically into $\mathcal{Q}$. A partially isometric transformation of a Krein space into a Krein space is contractive if, and only if, its kernel is a Hilbert space. Complementation is preserved under contractive partially isometric transformations of a Krein space onto a Krein space.

Theorem 5. *If a contractive partially isometric transformation T maps a Krein space $\mathcal{H}$ onto a Krein space $\mathcal{H}'$ and if Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained continuously and contractively as complementary subspaces of $\mathcal{H}$, then Krein spaces $\mathcal{P}'$ and $\mathcal{Q}'$, which are contained continuously and contractively as complementary subspaces of $\mathcal{H}'$, exist such that T acts as a contractive partially isometric transformation of $\mathcal{P}$ onto $\mathcal{P}'$ and of $\mathcal{Q}$ onto $\mathcal{Q}'$.*

Proof of Theorem 5. Since the Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained continuously and contractively in $\mathcal{H}$ and since T is a continuous and contractive transformation of $\mathcal{H}$ into $\mathcal{H}'$, T acts as a continuous and contractive transformation of $\mathcal{P}$ into $\mathcal{H}'$ and of $\mathcal{Q}$ into $\mathcal{H}'$. Krein spaces $\mathcal{P}'$ and $\mathcal{Q}'$, which are contained continuously and contractively in $\mathcal{H}'$, exist such that T acts as a contractive partially isometric transformation of $\mathcal{P}$ onto $\mathcal{P}'$ and of $\mathcal{Q}$ onto $\mathcal{Q}'$. It will be shown that $\mathcal{P}'$ and $\mathcal{Q}'$ are complementary subspaces of $\mathcal{H}'$.

An element a of $\mathcal{P}'$ is of the form Ta for an element a of $\mathcal{P}$ such that

$$\langle Ta, Ta\rangle_{\mathcal{P}'} = \langle a, a\rangle_{\mathcal{P}}.$$

An element b of $\mathcal{Q}'$ is of the form Tb for an element b of $\mathcal{Q}$ such that

$$\langle Tb, Tb\rangle_{\mathcal{Q}'} = \langle b, b\rangle_{\mathcal{Q}}.$$

The element $c = a + b$ of $\mathcal{H}$ satisfies the inequalities

$$\langle c, c\rangle_{\mathcal{H}} \leq \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

and

$$\langle Tc, Tc\rangle_{\mathcal{H}'} \leq \langle c, c\rangle_{\mathcal{H}}.$$

The element $Tc = Ta + Tb$ of $\mathcal{H}'$ satisfies the inequality

$$\langle Tc, Tc\rangle_{\mathcal{H}'} \leq \langle Ta, Ta\rangle_{\mathcal{P}'} + \langle Tb, Tb\rangle_{\mathcal{Q}'}.$$

An element of $\mathcal{H}'$ is of the form Tc for an element c of $\mathcal{H}$ such that

$$\langle Tc, Tc\rangle_{\mathcal{H}'} = \langle c, c\rangle_{\mathcal{H}}.$$

An element a of $\mathcal{P}$ and an element b of $\mathcal{Q}$ exist such that $c = a + b$ and

$$\langle c, c\rangle_{\mathcal{H}} = \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}.$$

Since the element Ta of $\mathcal{P}'$ satisfies the inequality

$$\langle Ta, Tb\rangle_{\mathcal{P}'} \leq \langle a, a\rangle_{\mathcal{P}}$$

and since the element Tb of $\mathcal{Q}'$ satisfies the inequality

$$\langle Tb, Tb\rangle_{\mathcal{Q}'} \leq \langle b, b\rangle_{\mathcal{Q}},$$

the element Tc of $\mathcal{H}$ satisfies the inequality

$$\langle Tc, Tc\rangle_{\mathcal{H}'} \geq \langle Ta, Ta\rangle_{\mathcal{P}'} + \langle Tb, Tb\rangle_{\mathcal{Q}'}.$$

Equality holds since the reverse inequality is satisfied.

This completes the proof of the theorem.

A canonical coisometric linear system whose state space is a Hilbert space is constructed when multiplication by $W(z)$ is a contractive transformation in $\mathcal{C}(z)$. The range of multiplication by $W(z)$ in $\mathcal{C}(z)$ is a Hilbert space which is contained contractively in $\mathcal{C}(z)$ when considered with the unique scalar product such that multiplication by $W(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the range. The complementary space in $\mathcal{C}(z)$ to the range is the state space $\mathcal{H}(W)$ of a canonical coisometric linear system with transfer function $W(z)$. Every Hilbert space which is the state space of a canonical coisometric linear system is so obtained.

A Herglotz space is a Hilbert space, whose elements are power series, such that the difference-quotient transformation is a continuous transformation of the space into itself which has an isometric adjoint and such that a continuous transformation of the space into the space of complex numbers is defined by taking $f(z)$ into $f(0)$. A continuous transformation of the space into the space of complex numbers is then defined by taking a power series into its coefficient of z^n for every nonnegative integer n. A Herglotz function for the space is a power series

$$\phi(z) = \sum \phi_n z^n$$

such that the adjoint of the continuous transformation of the space into the complex numbers takes a complex number c into

$$\tfrac{1}{2}[z^n \phi(z) + \phi_0^- z^n + \cdots + \phi_n^-]c.$$

A Herglotz function for the space is determined within an added imaginary constant by the adjoint computation when n is zero. The form of the adjoint for positive integers n is verified inductively using the isometric property of the adjoint of the difference-quotient transformation. The adjoint transformation takes $f(z)$ into $zf(z) + c$ for a vector c which depends continuously on $f(z)$ and which is computed inductively in the present application. A Herglotz space is uniquely determined by its Herglotz function.

The Herglotz function of a Herglotz space is a power series $\phi(z)$ which represents a function whose values in the unit disk have nonnegative real part. The Poisson representation of $\phi(z)$ is an integral

$$\frac{\phi(z) + \phi(w)^-}{1 - zw^-} = \frac{1}{\pi} \int \frac{d\mu(\theta)}{(1 - e^{-i\theta} z)(1 - e^{i\theta} w^-)}$$

with respect to a nonnegative measure μ on the Borel sets of the real numbers modulo 2π. The integral converges when z and w are in the unit disk. A unique Herglotz space $\mathcal{L}(\phi)$ exists which has $\phi(z)$ as Herglotz function. A continuous transformation of the space $L^2(\mu)$ onto the space $\mathcal{L}(\phi)$ exists which takes a function $f(\theta)$ of θ in the real numbers modulo 2π into the power series $g(z)$ with complex coefficients which satisfies the identity

$$2\pi c^- g(w) = \int c^- (1 - e^{-i\theta} w)^{-1} d\mu(\theta) f(\theta)$$

for every complex numbers c when w is in the unit disk. The transformation takes the function $e^{-i\theta} f(\theta)$ of θ in the real numbers modulo 2π into the power series $[g(z) - g(0)]/z$ whenever it takes the function $f(\theta)$ of θ into the power series $g(z)$. The identity

$$2\pi \langle g(z), g(z) \rangle_{\mathcal{L}(\phi)} = \int f(\theta)^- d\mu(\theta) f(\theta)$$

holds when the function $f(\theta)$ of θ in the real numbers modulo 2π is orthogonal to the kernel of the transformation.

The extension space $\operatorname{ext} \mathcal{L}(\phi)$ of the Herglotz space $\mathcal{L}(\phi)$ is a Hilbert space of Laurent series, which is invariant under division by z, such that the canonical projection onto the space $\mathcal{L}(\phi)$ is a partial isometry. The canonical projection takes a Laurent series into the power series which has the same coefficient of z^n for every nonnegative integer n. Uniqueness of the extension space results from the hypothesis that an element $f(z)$ of the space vanishes if the projection of $z^n f(z)$ in the space $\mathcal{L}(\phi)$ vanishes for every nonnegative integer n. The norm of $f(z)$ in the extension space is the least upper bound of the norms of the projections of $z^n f(z)$ in the given space. Division by z is an isometric transformation in the space $\operatorname{ext} \mathcal{L}(\phi)$ whose adjoint is isometric. A unique continuous transformation of the space $\mathcal{L}^2(\mu)$ onto the extension space exists whose composition with the canonical projection onto the given space is the continuous transformations onto the given space and which takes the function $e^{-i\theta}f(\theta)$ of θ in the real modulo 2π into the Laurent series $z^{-1}g(z)$ whenever it takes the function $f(\theta)$ of θ into the Laurent series $g(z)$. The identity

$$2\pi\langle g(z), g(z)\rangle_{\mathrm{ext}\,\mathcal{L}(\phi)} = \int f(\theta)^- d\mu(\theta) f(\theta)$$

holds whenever the transformation takes a function $f(\theta)$ of θ in the real numbers modulo 2π into the Laurent series $g(z)$.

A spectral subspace of contractivity is constructed for a closed relation T whose domain is contained in a Hilbert space $\mathcal{P}$ and whose range is contained in a Hilbert space $\mathcal{Q}$. The relation T is then the adjoint of the adjoint relation T^* which has its domain contained in the Hilbert space $\mathcal{Q}$ and its range contained in the Hilbert space $\mathcal{P}$. A self-adjoint relation H in the Cartesian product Hilbert space $\mathcal{P} \times \mathcal{Q}$ is defined by taking (a, b) into (T^*b, Ta) when a is in the domain of T and b is in the domain of T^*. The spectral subspace of contractivity for H is a closed subspace of the Cartesian product such that H acts as a contractive transformation of the subspace into itself. The orthogonal complement is a closed subspace such that the inverse of H acts as a contractive transformation of the subspace into itself. Eigenvectors of H for eigenvalues of absolute value one belong to the spectral subspace of contractivity for H. The square of H is a self-adjoint relation in the Cartesian product space which has the same spectral subspace of contractivity. Since the transformation which takes (a, b) into $(a, -b)$ commutes with the square of H, the spectral subspace of contractivity for H is the Cartesian product of a closed subspace of $\mathcal{P}$ and a closed subspace of $\mathcal{Q}$. The spectral subspace of contractivity for T is the closed subspace of $\mathcal{P}$. The spectral subspace of contractivity for T^* is the closed subspace of $\mathcal{Q}$. The relation T acts as a contractive transformation of the spectral subspace of contractivity for T into the spectral subspace of contractivity for T^*. The relation T^* acts as a contractive transformation of the spectral subspace of contractivity for T^* into the spectral subspace of contractivity for T. The inverse of T acts as a contractive transformation of the orthogonal complement of the spectral subspace of contractivity for T^* into the orthogonal complement of the spectral subspace of contractivity for T. The inverse of T^* acts as a contractive transformation of the orthogonal complement of the spectral subspace of contractivity for T into the orthogonal complement of the spectral subspace of contractivity for T^*. If a is an element of $\mathcal{P}$ and if b is an element of $\mathcal{Q}$ such that the identities

$$Ta = b$$

and

$$T^*b = a$$

are satisfied, then a belongs to the spectral subspace of contractivity for T and b belongs to the spectral subspace of contractivity for T^*.

A Herglotz space is associated with the transfer function $W(z)$ of a canonical coisometric linear system whose state space is a Hilbert space. Since multiplication by $W(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$, the adjoint of multiplication by $W(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$. The range of the adjoint is a Hilbert space which is contained contractively in $\mathcal{C}(z)$ when it is considered with the scalar product such that the adjoint acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the range. The space is a Herglotz space. The space $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1)$ whose Herglotz function is identically one. The complementary space to the range Herglotz space is a Herglotz space whose Herglotz function $\phi(z)$ is determined within an added constant, which is a skew-conjugate operator, by the identity

$$\phi(z) + \phi^*(z^{-1}) = 2 - 2W^*(z^{-1})W(z).$$

The space $\mathcal{L}(\phi)$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The scalar product in the space $\mathcal{L}(\phi)$ is determined by the identity

$$\langle f(z), f(z)\rangle_{\mathcal{L}(\phi)} = \langle f(z), f(z)\rangle_{\mathcal{C}(z)} + \langle W(z)f(z), W(z)f(z)\rangle_{\mathcal{H}(W)}.$$

The adjoint of multiplication by $W(z)$ as a transformation in $\mathcal{C}(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the complementary space $\mathcal{L}(1-\phi)$ to $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$. Since the polynomial elements of $\mathcal{C}(z)$ are dense in $\mathcal{C}(z)$, the polynomial elements of the space $\mathcal{L}(1-\phi)$ are dense in the space $\mathcal{L}(1-\phi)$.

If a Herglotz space $\mathcal{L}$ is contained contractively in $\mathcal{C}(z)$ and if the polynomial elements of the space are dense in the space, then a partially isometric transformation of $\mathcal{C}(z)$ onto $\mathcal{L}$ exists which commutes with the difference-quotient transformation and whose kernel is invariant under multiplication by z. The resulting contractive transformation of $\mathcal{C}(z)$ into itself coincides with the adjoint of multiplication by $V(z)$ for a power series $V(z)$ with complex coefficients and that multiplication by $V(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$ and such that the orthogonal complement of the range of multiplication by $V(z)$ in $\mathcal{C}(z)$ is invariant under multiplication by z.

The construction of $V(z)$ is supplied when $W(z)$ is a power series such that multiplication by $W(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$. The associated Herglotz space $\mathcal{L}(\phi)$ contains the elements $f(z)$ of $\mathcal{C}(z)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(W)$. The identity

$$\|f(z)\|^2_{\mathcal{L}(\phi)} = \|f(z)\|^2_{\mathcal{C}(z)} + \|W(z)f(z)\|^2_{\mathcal{H}(W)}$$

holds for every element $f(z)$ of the space $\mathcal{L}(\phi)$. The space $\mathcal{L}(\phi)$ is contained contractively in $\mathcal{C}(z)$. The complementary space to $\mathcal{L}(\phi)$ in $\mathcal{C}(z)$ is a Herglotz space $\mathcal{L}(1-\phi)$, which is contained contractively in $\mathcal{C}(z)$, such that the adjoint of multiplication by $W(z)$ in $\mathcal{C}(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(1-\phi)$. The polynomial elements of the space $\mathcal{L}(1-\phi)$ are dense in the space. A power series $V(z)$, such that multiplication by $V(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$, exists such that the adjoint of multiplication by $V(z)$ as a transformation in $\mathcal{C}(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the space $\mathcal{L}(1-\phi)$ and such that the kernel of the adjoint transformation is invariant under multiplication by z. A power series $U(z)$ exists such that multiplication by $U(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$, such that the adjoint of multiplication by $W(z)$ in $\mathcal{C}(z)$ is the composition of the adjoint of multiplication by $U(z)$ in $\mathcal{C}(z)$ and the adjoint of multiplication by $V(z)$ in $\mathcal{C}(z)$, and such that the range of the adjoint of multiplication by $U(z)$ in $\mathcal{C}(z)$ is orthogonal to the kernel of the adjoint of multiplication by $V(z)$ in $\mathcal{C}(z)$.

The Beurling factorization

$$W(z) = V(z)U(z)$$

results of a power series $W(z)$ such that multiplication by $W(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$. The outer function $V(z)$ is a power series such that multiplication by $V(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$ and such that the orthogonal complement of the kernel of multiplication by $V(z)$ is invariant under multiplication by z. The inner function $U(z)$ is a power series such that multiplication by $U(z)$ is a partially isometric transformation in $\mathcal{C}(z)$ and such that the range of multiplication by $U(z)$ in $\mathcal{C}(z)$ is orthogonal to the kernel of multiplication by $V(z)$.

The Nevanlinna factorization of a power series $W(z)$, such that multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$, is a variant of the Beurling factorization. An outer function is again a power series $V(z)$ such that multiplication by $V(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$ and such that the orthogonal complement of the range of multiplication by $V(z)$ is invariant under multiplication by z.

Theorem 6. *If $W(z)$ is a power series such that multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$, then an outer function $V(z)$ exists such that multiplication by*

$$U(z) = W(z)V(z)$$

is an everywhere defined and contractive transformation in $\mathcal{C}(z)$ and such that no nonzero element $f(z)$ of the space $\mathcal{H}(V)$ exists such that $W(z)f(z)$ belongs to the space $\mathcal{H}(U)$.

Proof of Theorem 6. Multiplication by $W(z)$ is extended to a transformation with domain and range in ext $\mathcal{C}(z)$ which commutes with multiplication by z. An element $f(z)$ of the space ext $\mathcal{C}(z)$ belongs to the domain of multiplication by $W(z)$ as transformation in ext $\mathcal{C}(z)$ if $z^r f(z)$ belongs to the domain of multiplication by $W(z)$ in $\mathcal{C}(z)$ for some nonnegative integer r. Multiplication by $W(z)$ in ext $\mathcal{C}(z)$ takes $z^r f(z)$ into $z^r g(z)$ when multiplication by $W(z)$ in $\mathcal{C}(z)$ takes $f(z)$ into $g(z)$. The definition is independent of r. Multiplication by $W(z)$ in ext $\mathcal{C}(z)$ is a densely defined transformation in ext $\mathcal{C}(z)$, whose closure is however not assumed to be a transformation. The adjoint of multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$ is a transformation. The spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ in ext $\mathcal{C}(z)$ and its orthogonal complement are invariant subspaces for multiplication and division by z.

The space ext $\mathcal{C}(z)$ is contained contractively in a Hilbert space ext $\mathcal{P}$ such that a dense set of elements of the complementary space to ext $\mathcal{C}(z)$ in ext $\mathcal{P}$ belong to ext $\mathcal{C}(z)$ and such that the adjoint of multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$ maps the intersection of its domain with the orthogonal complement of its spectral subspace of contractivity onto the intersection of ext $\mathcal{C}(z)$ with the complementary space to ext $\mathcal{C}(z)$ in ext $\mathcal{P}$. The intersection of ext $\mathcal{C}(z)$ with the complementary space to ext $\mathcal{C}(z)$ in ext $\mathcal{P}$ is invariant under division by z. Division by z is an isometric transformation with respect to the scalar product of $\mathcal{P}$ as well as with respect to the scalar product of the complementary space to ext $\mathcal{C}(z)$ in ext $\mathcal{P}$.

The canonical projection of ext $\mathcal{C}(z)$ onto $\mathcal{C}(z)$ determines a partially isometric transformation of ext $\mathcal{P}$ onto a Hilbert space $\mathcal{P}$, a dense set of whose elements belong to $\mathcal{C}(z)$. The space $\mathcal{C}(z)$ is contained contractively in the space $\mathcal{P}$. The partially isometric transformation of ext $\mathcal{P}$ onto $\mathcal{P}$ acts as a partially isometric transformation of the complementary space to ext $\mathcal{C}(z)$ in ext $\mathcal{P}$ onto the complementary space to $\mathcal{C}(z)$ in $\mathcal{P}$. The intersection of $\mathcal{C}(z)$ with $\mathcal{P}$ and the intersection of $\mathcal{C}(z)$ with the complementary space to $\mathcal{C}(z)$ in $\mathcal{P}$ are invariant subspaces for the difference-quotient transformation. The continuous extension of the difference-quotient transformation has an isometric adjoint in $\mathcal{P}$ as well as in the complementary space to $\mathcal{C}(z)$ in $\mathcal{P}$.

Since the polynomial elements of $\mathcal{C}(z)$ are dense in $\mathcal{P}$, an isometric transformation of $\mathcal{P}$ onto $\mathcal{C}(z)$ exists which intertwines the continuous extension of the difference-quotient transformation in $\mathcal{P}$ with the difference-quotient transformation in $\mathcal{C}(z)$. Since $\mathcal{C}(z)$ is contained contractively in $\mathcal{P}$, a contractive transformation of $\mathcal{C}(z)$ into itself is obtained which commutes with the difference-quotient transformation. The transformation is the adjoint of multiplication by $V(z)$ for a power series $V(z)$ such that multiplication by $V(z)$ is everywhere defined and contractive as a transformation in $\mathcal{C}(z)$. A Hilbert space $\mathcal{H}(V)$ exists which is the state space of a canonical coisometric linear system with transfer function $V(z)$. The Herglotz space $\mathcal{L}(\phi)$ associated with the space $\mathcal{H}(V)$ is contained contractively in $\mathcal{C}(z)$. The continuous extension of the adjoint of multiplication by $V(z)$ acts as an isometric transformation of $\mathcal{P}$ onto $\mathcal{C}(z)$. The adjoint of multiplication by $V(z)$ as a transformation in $\mathcal{C}(z)$ acts as an isometric transformation of $\mathcal{C}(z)$ onto the complementary space $\mathcal{L}(1-\psi)$ to the space $\mathcal{L}(\psi)$ in $\mathcal{C}(z)$. The continuous extension of the adjoint of multiplication by $V(z)$ as a transformation in $\mathcal{C}(z)$ acts as an isometric transformation of the complementary space to $\mathcal{C}(z)$ in $\mathcal{P}$ onto the space $\mathcal{L}(\psi)$.

A Hilbert space ext $\mathcal{Q}$, which is contained contractively in the space ext $\mathcal{P}$, exists such that the intersection of ext $\mathcal{Q}$ with ext $\mathcal{C}(z)$ is the range of the adjoint of multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$. The space ext $\mathcal{Q}$ is the orthogonal sum of its intersection with the spectral subspace of contractivity for multiplication by $W(z)$ in ext $\mathcal{C}(z)$ and the closure of its intersection with the orthogonal complement in ext $\mathcal{C}(z)$ of the spectral subspace. The complementary space to ext $\mathcal{C}(z)$ in ext $\mathcal{Q}$ is isometrically equal to the closure in ext $\mathcal{Q}$ of its intersection with the orthogonal complement of the spectral subspace. The adjoint of multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$ acts as a partially isometric transformation of its spectral subspace of contractivity onto the intersection of ext $\mathcal{Q}$ with the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$. The space ext $\mathcal{Q}$ and its complementary space in the space ext $\mathcal{P}$ are invariant subspaces for the continuous extension of division by z. The continuous extension of division by z is an isometric transformation in ext $\mathcal{Q}$ and its complementary space in ext $\mathcal{P}$.

The partially isometric transformation of ext $\mathcal{P}$ onto $\mathcal{P}$, which is determined by the canonical projection of ext $\mathcal{C}(z)$ onto $\mathcal{C}(z)$, acts as a partially isometric transformation of ext $\mathcal{Q}$ onto a Hilbert space which is contained contractively in $\mathcal{P}$. The canonical projection of ext $\mathcal{C}(z)$ onto $\mathcal{C}(z)$ acts as a partially isometric transformation of the complementary space to ext $\mathcal{Q}$ in ext $\mathcal{P}$ onto the complementary space to $\mathcal{Q}$ in $\mathcal{P}$. The space $\mathcal{Q}$ and its complementary space in $\mathcal{P}$ are invariant subspaces for the continuous extension of the difference-quotient transformation. The continuous extension of the difference-quotient transformation has an isometric adjoint in $\mathcal{Q}$ and in its complementary space in $\mathcal{P}$.

The power series
$$U(z) = W(z)V(z)$$
has properties which are derived from adjoints of multiplication transformations. Since multiplication by $W(z)$ is a densely defined transformation in $\mathcal{C}(z)$ by hypothesis, the adjoint of multiplication by $W(z)$ as a transformation in $\mathcal{C}(z)$ is contained in the closure of the composition of the adjoint of multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$ with the canonical projection of ext $\mathcal{C}(z)$ onto $\mathcal{C}(z)$. The range of the adjoint of multiplication by $W(z)$ as a transformation in $\mathcal{C}(z)$ is contained in $\mathcal{Q}$. The continuous extension of the adjoint of multiplication by $W(z)$ as a transformation in $\mathcal{C}(z)$ is a contractive transformation of $\mathcal{C}(z)$ into $\mathcal{Q}$. The continuous extension of the adjoint of multiplication by $V(z)$ as a transformation in $\mathcal{C}(z)$ acts as a contractive transformation of $\mathcal{Q}$ into $\mathcal{C}(z)$. Since the adjoint of multiplication by $U(z)$ in $\mathcal{C}(z)$ is contained in the composition of the continuous extension of the adjoint of multiplicative by $W(z)$ in $\mathcal{C}(z)$ with the continuous extension of the adjoint of multiplication by $V(z)$ in $\mathcal{C}(z)$ and since $\mathcal{C}(z)$ is contained contractively in $\mathcal{Q}$, the adjoint of multiplication by $U(z)$ is an everywhere defined and contractive transformation in $\mathcal{C}(z)$. The contractive property is first verified on polynomial elements of $\mathcal{C}(z)$. If then follows for all elements of $\mathcal{C}(z)$. Multiplication by $U(z)$ is everywhere defined and contractive as a transformation in $\mathcal{C}(z)$.

A Hilbert space $\mathcal{H}(U)$ exists which is the state space of a canonical coisometric linear system with transfer function $U(z)$. The adjoint of multiplication by $U(z)$ as a transformation in $\mathcal{C}(z)$ acts as a partially isometric transformation of $\mathcal{C}(z)$ onto the complementary space $\mathcal{L}(1-\phi)$ in $\mathcal{C}(z)$ to the Herglotz space $\mathcal{L}(\phi)$ associated with the space $\mathcal{H}(0)$.

Since the complementary space to $\mathcal{C}(z)$ in the space $\mathcal{P}$ is contained isometrically in the space $\mathcal{Q}$, no nonzero element of the complementary space to $\mathcal{C}(z)$ in the space $\mathcal{P}$ belongs to the complementary space to the space $\mathcal{Q}$ in the space $\mathcal{P}$. Since the continuous extension of the adjoint of multiplication by $V(z)$ in $\mathcal{C}(z)$ acts as an isometric transformation of the complementary space to $\mathcal{C}(z)$ in $\mathcal{P}$ onto the space $\mathcal{L}(\psi)$ and of the complementary space to $\mathcal{Q}$ in $\mathcal{P}$ onto $\mathcal{L}(\phi)$, the intersection of the spaces $\mathcal{L}(\phi)$ and $\mathcal{L}(\psi)$ contains no nonzero element.

The space $\mathcal{H}(V)$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that the adjoint of multiplication by $V(z)$ in $\mathcal{C}(z)$ maps $f(z)$ into an element $g(z)$ of the space $\mathcal{L}(\psi)$. The identity
$$\|f(z)\|^2_{\mathcal{H}(V)} = \|f(z)\|^2_{\mathcal{C}(z)} + \|g(z)\|^2_{\mathcal{L}(\psi)}$$
is then satisfied. Since the continuous extension of the adjoint of multiplication by $V(z)$ as a transformation in $\mathcal{C}(z)$ acts as an isometric transformation of the complementary space

to $\mathcal{C}(z)$ in the space $\mathcal{P}$ onto the space $\mathcal{L}(\psi)$, the space $\mathcal{H}(V)$ is the intersection of $\mathcal{C}(z)$ with the complementary space to $\mathcal{C}(z)$ in the space $\mathcal{P}$. The square of the norm of an element of the space $\mathcal{H}(V)$ is the sum of the square of its norm as an element of $\mathcal{C}(z)$ and the square of its norm as an element of the complementary space to $\mathcal{C}(z)$ in the space $\mathcal{P}$.

The space $\mathcal{H}(U)$ is the set of element $f(z)$ of $\mathcal{C}(z)$ such that the adjoint of multiplication by $U(z)$ in $\mathcal{C}(z)$ maps $f(z)$ into an element $g(z)$ of the space $\mathcal{L}(\phi)$. The identity

$$\|f(z)\|^2_{\mathcal{H}(U)} = \|f(z)\|^2_{\mathcal{C}(z)} + \|g(z)\|^2_{\mathcal{L}(\phi)}$$

is then satisfied. Since the continuous extension of the adjoint of multiplication by $V(z)$ as a transformation in $\mathcal{C}(z)$ acts as an isometric transformation of the complementary space to $\mathcal{Q}$ in $\mathcal{P}$ onto the space $\mathcal{L}(\phi)$, the space $\mathcal{H}(U)$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that the adjoint of multiplication by $W(z)$ in $\mathcal{C}(z)$ maps $f(z)$ into an element $h(z)$ of the complementary space to $\mathcal{Q}$ in $\mathcal{P}$. Since $h(z)$ then belongs to the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$, the element $f(z)$ of $\mathcal{C}(z)$ is the projection of an element of ext $\mathcal{C}(z)$ which belongs to the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$.

If $f(z)$ is an element of the space $\mathcal{H}(V)$ such that $W(z)f(z)$ belongs to the space $\mathcal{H}(U)$, then $W(z)f(z)$ belongs to the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$ since $f(z)$ belongs to the orthogonal complement of the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$. An element $g(z)$ of the spectral subspace of contractivity for the adjoint of multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$ exists which has $W(z)f(z)$ as its orthogonal projection in $\mathcal{C}(z)$. The product $W(z)f(z)$ is equal to zero since it is orthogonal to $g(z)$. The element $f(z)$ of $\mathcal{C}(z)$ is equal to zero since it is orthogonal to the spectral subspace of contractivity for multiplication by $W(z)$ as a transformation in ext $\mathcal{C}(z)$.

This completes the proof of the theorem.

The Hilbert space $\mathcal{H}(U)$ is contained continuously and isometrically in a Krein space $\mathcal{H}(W)$, whose elements are power series, such that multiplication by $W(z)$ acts as an isometric transformation of the anti-space of the Hilbert space $\mathcal{H}(V)$ onto the orthogonal complement of the space $\mathcal{H}(U)$ in the space $\mathcal{H}(W)$. The space $\mathcal{H}(U')$ corresponding to the power series

$$U'(z) = zU(z)$$

with complex coefficients is the set of power series $f(z)$ with complex coefficients such that $[f(z) - f(0)]/z$ belongs to the space $\mathcal{H}(U)$. The identity for difference quotients

$$\|[f(z) - f(0)]/z\|^2_{\mathcal{H}(U)} = \|f(z)\|^2_{\mathcal{H}(U')} - f(0)^- f(0)$$

is satisfied. The space $\mathcal{H}(U)$ is contained contractively in the space $\mathcal{H}(U')$. The space $\mathcal{H}(W')$ corresponding to the power series

$$W'(z) = zW(z)$$

with complex coefficients is the set of power series $f(z)$ with complex coefficients such that $[f(z) - f(0)]/z$ belongs to the space $\mathcal{H}(W)$. The identity for difference quotients

$$\langle [f(z) - f(0)]/z, \ [f(z) - f(0)]/z \rangle_{\mathcal{H}(W)} = \langle f(z), f(z) \rangle_{\mathcal{H}(W')} - f(0)^- f(0)$$

is satisfied. The space $\mathcal{H}(U')$ is contained continuously and isometrically in the space $\mathcal{H}(W')$. Multiplication by $W'(z)$ is an isometric transformation of the anti-space of the Hilbert space $\mathcal{H}(V)$ onto the orthogonal complement of the space $\mathcal{H}(U')$ in the space $\mathcal{H}(W')$. The space $\mathcal{H}(W)$ is contained continuously and contractively in the space $\mathcal{H}(W')$. Multiplication by $W(z)$ is a partially isometric transformation of the space of complex numbers onto the complementary space to the space $\mathcal{H}(W)$ in the space $\mathcal{H}(W')$. The space $\mathcal{H}(W)$ is the state space of a canonical coisometric linear system with transfer function $W(z)$.

A canonical unitary linear is constructed from a canonical coisometric linear system with state space $\mathcal{H}(W)$ and transfer function $W(z)$ when the inequality for difference quotients

$$\langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}(W)} \leq \langle f(z), f(z) \rangle_{\mathcal{H}(W)} - f(0)^- f(0)$$

holds for every element $f(z)$ of the space. The inequality is satisfied by the canonical coisometric linear system constructed when multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$. The elements of the state space $\mathcal{D}(W)$ of the canonical unitary linear system are pairs $(f(z), g(z))$ of power series. Power series $f(z)$ and

$$g(z) = \sum a_n z^n$$

with complex coefficients determine an element of the space $\mathcal{D}(W)$ if $f(z)$ is an element of the space $\mathcal{H}(W)$ such that

$$z^{r+1} f(z) - W(z)(a_0 z^r + \ldots + a_r)$$

belongs to the space $\mathcal{H}(W)$ for every nonnegative integer r and such that the sequence of numbers

$$\langle z^{r+1} f(z) - W(z)(a_0 z^r + \ldots + a_r), z^{r+1} f(z) - W(z)(a_0 z^r + \ldots + a_r) \rangle_{\mathcal{H}(W)} + a_0^- a_0 + \ldots + a_r^- a_r$$

is bounded. The inequality for difference quotients in the space $\mathcal{H}(W)$ implies that the sequence is nondecreasing. The limit of the sequence is taken as the definition of the scalar self–product

$$\langle f(z), g(z)), (f(z), g(z)) \rangle_{\mathcal{D}(W)}.$$

The space $\mathcal{D}(W)$ is a Krein space. A contractive partially isometric transformation of the space $\mathcal{D}(W)$ onto the space $\mathcal{H}(W)$ is defined by taking $(f(z), g(z))$ into $f(z)$. A continuous transformation of the space $\mathcal{D}(W)$ into itself is defined by taking $(f(z), g(z))$ into

$$([f(z) - f(0)]/z, zg(z) - W^*(z) f(0)).$$

The identity for difference quotients

$$\langle([f(z)-f(0)]/z, zg(z)-W^*(z)f(0)),\ ([f(z)-f(0)]/z, zg(z)-W^*(z)f(z))\rangle_{\mathcal{D}(W)}$$
$$= \langle(f(z),g(z)),(f(z),g(z))\rangle_{\mathcal{D}(W)} - f(0)^-f(0)$$

is satisfied. The adjoint transformation of the space $\mathcal{D}(W)$ into itself takes $(f(z), g(z))$ into

$$(zf(z)-W(z)g(0), [g(z)-g(0)]/z).$$

The identity for difference quotients

$$\langle(zf(z)-W(z)g(0), [g(z)-g(0)]/z),\ (zf(z)-W(z)g(0), [g(z)-g(0)]/z)\rangle_{\mathcal{D}(W)}$$
$$= \langle(f(z),g(z)),(f(z),g(z))\rangle_{\mathcal{D}(W)} - g(0)^-g(0)$$

is satisfied.

A construction has been made of the state space $\mathcal{D}(W)$ of a canonical unitary linear system with transfer function $W(z)$. The main transformation takes $(f(z), g(z))$ into

$$([f(z)-f(0)]/z, zg(z)-W^*(z)f(0)).$$

The input transformation takes c into

$$([W(z)-W(0)]c/z, [1-W^*(z)W(0)]c).$$

The output transformation takes $(f(z), g(z))$ into $f(0)$. The external operator is $W(0)$. The unitary property of the linear system is a consequence of the two identities for difference quotients. The transformation which takes $(f(z), g(z))$ into $(g(z), f(z))$ maps the space $\mathcal{D}(W)$ isometrically onto the state space $\mathcal{D}(W^*)$ of a canonical unitary linear system with transfer function $W^*(z)$.

Uniqueness of a canonical unitary linear system with transfer function $W(z)$ is obtained when multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$.

Theorem 7. *If $W(z)$ is a power series such that multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$, if $V(z)$ and*

$$U(z) = W(z)V(z)$$

are power series such that multiplication by $U(z)$ and multiplication by $V(z)$ are everywhere defined and contractive as transformations in $\mathcal{C}(z)$, if no nonzero element $f(z)$ of the space $\mathcal{H}(V)$ exists such that $W(z)f(z)$ belongs to the space $\mathcal{H}(U)$, and if $\mathcal{D}(W)$ is the state space of a canonical unitary linear system with transfer function $W(z)$, then a contractive partially isometric transformation of the space $\mathcal{D}(W)$ onto a Krein space $\mathcal{H}(W)$, which is the state space of a canonical coisometric linear system with transfer function $W(z)$, is defined by taking $(f(z), g(z))$ into $f(z)$. The space $\mathcal{H}(U)$ is contained continuously and isometrically in the space $\mathcal{H}(W)$. Multiplication by $W(z)$ is a partially isometric *transformation of the anti–space of the space $\mathcal{H}(V)$ onto the orthogonal complement of the space $\mathcal{H}(U)$ in the space $\mathcal{H}(W)$.*

Proof of Theorem 7. A transformation of the Cartesian product of the space $\mathcal{D}(W)$ and the space $\mathcal{D}(V)$ onto a vector space $\mathcal{D}$, whose elements are pairs of power series with complex coefficients, is defined by taking an element $(f(z), g(z))$ of the space $\mathcal{D}(W)$ and an element $(h(z), k(z))$ of the space $\mathcal{D}(V)$ into the element $(u(z), v(z))$ of the space $\mathcal{D}$ defined by

$$u(z) = f(z) + W(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z).$$

The space $\mathcal{D}$ is the state space of a linear system with transfer function $U(z)$. The main transformation takes $(u(z), v(z))$ into

$$([u(z) - u(0)]/z, zv(z) - U^*(z)u(0)).$$

If

$$u(z) = f(z) + W(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z),$$

then

$$[u(z) - u(0)]/z = f'(z) + W(z)h'(z)$$

and

$$zv(z) - U^*(z)u(0) = k'(z) + V^*(z)g'(z)$$

with $(f'(z), g'(z))$ the element of the space $\mathcal{D}(W)$ defined by

$$f'(z) = [f(z) - f(0)]/z + [W(z) - W(0)]h(0)/z$$

and

$$g'(z) = zg(z) - W^*(z)h(0) + [1 - W^*(z)W(0)]h(0)$$

and with $(h'(z), k'(z))$ the element of the space $\mathcal{D}(V)$ defined by

$$h'(z) = [h(z) - h(0)]/z$$

and

$$k'(z) = zk(z) - V^*(z)h(0).$$

The input transformation takes a complex number c into

$$([1 - U(z)U(0)^-]c, [U^*(z) - U^*(0)]c/z)$$

where

$$[1 - U(z)U(0)^-]c = [1 - W(z)W(0)^-]c + W(z)[1 - V(z)V(0)^-]W(0)^-c$$

and

$$[U^*(z) - U^*(0)]c/z = [V^*(z) - V^*(0)]W^*(0)c/z + V^*(z)[U^*(z) - U^*(0)]c/z$$

with

$$([1 - W(z)W(0)^-]c, [W^*(z) - W^*(0)]c/z)$$

an element of the space $\mathcal{D}(W)$ and

$$([1 - V(z)V(0)^-]W(0)^-c, [V^*(z) - V^*(0)]W^*(0)c/z)$$

an element of the space $\mathcal{D}(W)$. The output transformation takes $(u(z), v(z))$ into $u(0)$. If

$$u(z) = f(z) + W(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z)$$

with $(f(z), g(z))$ in the space $\mathcal{D}(W)$ and $(h(z), k(z))$ in the space $\mathcal{D}(V)$, then

$$u(0) = f(0) + W(0)g(0).$$

The external operator is

$$U(0) = W(0)V(0).$$

The matrix entries of the linear system with state space $\mathcal{D}$ and transfer function $U(z)$ are constructed from the matrix entries of a unitary linear system whose state space is the Cartesian product of the space $\mathcal{D}(W)$ and the space $\mathcal{D}(V)$ and whose transfer function is $U(z)$. A partially isometric transformation exists of the Cartesian product of the spaces $\mathcal{D}(W)$ and $\mathcal{D}(V)$ onto the space $\mathcal{D}(U)$ which is compatible with the structure of these spaces as state spaces of unitary linear systems with transfer function $U(z)$. Since the transformation of the Cartesian product space onto the space $\mathcal{D}$ is identical with the transformation of the Cartesian product space onto the space $\mathcal{D}(U)$, the space $\mathcal{D}$ is a Hilbert space which is the state space $\mathcal{D}(U)$ of a canonical unitary linear system with transfer function $U(z)$. The transformation of the Cartesian product space onto the space $\mathcal{D}$, equal to $\mathcal{D}(U)$, is a partial isometry.

A Krein space $\mathcal{E}$ is constructed whose elements are the pairs $(f(z), g(z))$ of power series such that

$$(-f(z), V^*(z)g(z))$$

belongs to the space $\mathcal{D}(V)$ and

$$(W(z)f(z), -g(z))$$

belongs to the space $\mathcal{D}(W)$. The scalar product is defined in the space so that the identity

$$\begin{aligned}\langle (f(z), g(z)), (f(z), g(z))\rangle_{\mathcal{E}} &= \langle (-f(z), V^*(z)g(z)), (-f(z), V^*(z)g(z))\rangle_{\mathcal{D}(V)} \\ &+ \langle (W(z)f(z), -g(z)), (W(z)f(z), -g(z))\rangle_{\mathcal{D}(W)}\end{aligned}$$

is satisfied. An isometric transformation of the space $\mathcal{E}$ onto itself is defined by taking $(f(z), g(z))$ into

$$([f(z) - f(0)]/z, f(0) + zg(z)).$$

The inverse isometric transformation takes $(f(z), g(z))$ into

$$(g(0) + zf(z), [g(z) - g(0)]/z).$$

An inverse isometric transformation exists of the space $\mathcal{D}(U)$ into the Cartesian product of the spaces $\mathcal{D}(W)$ and the space $\mathcal{D}(V)$. Every element of the space $\mathcal{D}(U)$ is uniquely of the form

$$(f(z) + W(z)h(z), k(z) + V^*(z)g(z))$$

for elements $(f(z), g(z))$ of the space $\mathcal{D}(W)$ and $(h(z), k(z))$ of the space $\mathcal{D}(V)$ such that the identity

$$\langle (f(z), g(z)), (W(z)u(z), -v(z))\rangle_{\mathcal{D}(W)} = \langle (h(z), k(z)), (u(z), -V^*(z)v(z))\rangle_{\mathcal{D}(V)}$$

holds for every element $(u(z), v(z))$ of the space $\mathcal{E}$. The image of the space $\mathcal{E}$ in the Cartesian product of the space $\mathcal{D}(W)$ and the space $\mathcal{D}(V)$ consists of the pairs of elements $(W(z)u(z), -v(z))$ of the space $\mathcal{D}(W)$ and elements $(-u(z), V^*(z)v(z))$ of the space $\mathcal{D}(V)$ which are parametrized by elements $(u(z), v(z))$ of the space $\mathcal{E}$. The pair of elements $(f(z), g(z))$ of the space $\mathcal{D}(W)$ and $(h(z), k(z))$ of the space $\mathcal{D}(V)$ is orthogonal in the Cartesian product space to the image of the space $\mathcal{E}$.

If $(h(z), k(z))$ is an element of the space $\mathcal{D}(V)$, an element $(u(z), v(z))$ of the space $\mathcal{E}$ exists such that an element of the Cartesian product space in the image of $\mathcal{D}(U)$ is obtained as the pair consisting of the element $(W(z)u(z), -v(z))$ of the space $\mathcal{D}(W)$ and the element

$$(h(z) - u(z), k(z) + V^*(z)v(z))$$

of the space $\mathcal{D}(V)$. Then

$$(W(z)[h(z) - u(z)], k(z) + V^*(z)v(z))$$

is an element of the space $\mathcal{D}(U)$. Since no nonzero element $f(z)$ of the space $\mathcal{H}(V)$ exists such that $W(z)f(z)$ belongs to the space $\mathcal{H}(U)$, no nonzero element $(f(z), g(z))$ of the space $\mathcal{D}(V)$ exists such that

$$(W(z)f(z), g(z))$$

belongs to the space $\mathcal{D}(U)$. It follows that

$$(h(z), k(z)) = (u(z), -V^*(z)v(z)).$$

The image of the space $\mathcal{D}(U)$ in the Cartesian product space consists of pairs of elements $(f(z), g(z))$ of the space $\mathcal{D}(W)$ such that $(f(z), V^*(z)g(z))$ belongs to the space $\mathcal{D}(U)$ and the zero element of the space $\mathcal{D}(V)$. A continuous isometric transformation of the space $\mathcal{D}(U)$ into the space $\mathcal{D}(W)$ is defined by taking $(f(z), V^*(z)g(z))$ into $(f(z), g(z))$. The orthogonal complement in the space $\mathcal{D}(W)$ of the image of the space $\mathcal{D}(U)$ consists of the elements of the space $\mathcal{D}(W)$ of the form $(W(z)u(z), -v(z))$ with $(u(z), v(z))$ in the space $\mathcal{E}$. Since $(-u(z), V^*(z)v(z))$ then belongs to the space $\mathcal{D}(V)$, the identity

$$-\langle (W(z)u(z), -v(z)), (W(z)u(z), -v(z))\rangle_{\mathcal{D}(W)}$$
$$= \langle (-u(z), V^*(z)v(z)), (-u(z), V^*(z)v(z))\rangle_{\mathcal{D}(V)}$$

is then satisfied. An isometric transformation of the anti–space of the space $\mathcal{D}(V)$ onto the orthogonal complement in the space $\mathcal{D}(W)$ of the image of the space $\mathcal{D}(U)$ is defined by taking $(u(z), -V^*(z)v(z))$ into $(W(z)u(z), -v(z))$.

The space $\mathcal{D}(W)$ is isometrically equal to the state space of the canonical unitary linear system with transfer function $W(z)$ which is constructed from the state space $\mathcal{H}(W)$ of the canonical coisometric linear system with transfer function $W(z)$ when multiplication by $W(z)$ is densely defined as a transformation in $\mathcal{C}(z)$. A contractive partially isometric transformation of the space $\mathcal{D}(W)$ onto the space $\mathcal{H}(W)$ is defined by taking $(f(z), g(z))$ into $f(z)$.

This completes the proof of the theorem.

The factorization theory of power series which represent functions analytic in the unit disk is treated when the Nevanlinna factorization theory does not apply. The multiplication of power series is considered only when they are the transfer functions of canonical unitary linear systems which are related to each other by the extension space of a Herglotz space.

Assume that $\mathcal{D}(U)$ is the state space of a canonical unitary linear system with transfer function $U(z)$ and that $\mathcal{D}(V)$ is the state space of a canonical unitary linear system with transformation $V(z)$ such that Hilbert space $\mathcal{C}$ of pairs of a power series with complex coefficients is constructed from the spaces $\mathcal{D}(U)$ and $\mathcal{D}(V)$. A pair $(f(z), g(z))$ of power series with complex coefficients belongs to the space ξ if, and only if,

$$(U(z)f(z), g(z))$$

belongs to the space $\mathcal{D}(U)$ and

$$(-f(z), V^*(z)g(z))$$

belongs to the space $\mathcal{D}(V)$. The identity

$$\langle (f(z), g(z)), (f(z), g(z))\rangle_{\mathcal{C}}$$
$$= \langle (U(z)f(z), -g(z)),\ (U(z)f(z), -g(z))\rangle_{\mathcal{D}(U)}$$
$$+\langle (-f(z), V^*(z)g(z)), (-f(z), V^*(z)g(z))\rangle_{\mathcal{D}(V)}$$

holds for every element $(f(z), g(z))$ of the space ξ. Then the state space $\mathcal{D}(W)$ of a canonical unitary linear system with transfer function

$$W(z) = U(z)V(z)$$

exists whose elements are pairs $(u(z), v(z))$ with

$$u(z) = f(z) + U(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z)$$

for elements $(f(z), g(z))$ of the space $\mathcal{D}(U)$ and $(h(z), k(z))$ of the space $\mathcal{D}(V)$. The scalar self–product

$$\langle (u(z), v(z)),\ (u(z), v(z))\rangle_{\mathcal{D}(W)}$$

is the greatest lower bound of sums

$$\langle (f(z), g(z)),\ (f(z), g(z))\rangle_{\mathcal{D}(U)}$$
$$+\langle (h(z), k(z)),\ (h(z), k(z))\rangle_{\mathcal{D}(V)}$$

of scalar self–products of representing elements $(f(z), g(z)$ of the space $\mathcal{D}(U)$ and $(h(z), k(z))$ of the space $\mathcal{D}(V)$.

If neither $U(\lambda)$ nor $V(\lambda)$ has absolute value one for some λ in the unit disk, then the set of elements $(f(z), g(z))$ of the space $\mathcal{D}(U)$ such that $f(z)$ vanishes at λ is a Krein space which is contained continuously and isometrically in the space $\mathcal{D}(U)$, and the set of elements $(f(z), g(z))$ of the space $\mathcal{D}(V)$ such that $f(z)$ vanishes at λ is a Krein space which is contained continuously and isometrically in the space $\mathcal{D}(V)$. A partially isometric transformation of the space $\mathcal{D}(U)$ onto a Krein space of dimension one is defined by taking $(f(z), g(z))$ into $f(\lambda)$. A partially isometric transformation of the space $\mathcal{D}(V)$ onto a Krein space of dimension one is defined by taking $(f(z), g(z))$ into $f(\lambda)$. The element

$$[1 - U(\lambda)U(\lambda)^-]/(1 - \lambda\lambda)^-)$$

of the image of the space $\mathcal{D}(U)$ has scalar self–product equal to itself. The element

$$[1 - V(\lambda)V(\lambda)^-]/(1 - \lambda\lambda^-)$$

of the image of the space $\mathcal{D}(V)$ has scalar product equal to itself. A partially isometric transformation of the space $\mathcal{E}$ onto a Hilbert space of dimension at most one is defined by taking $(f(z), g(z))$ into $f(\lambda)$. Since the scalar self–product of the element

$$\frac{1 - U(\lambda)U(\lambda)^-}{1 - \lambda\lambda^-}\ U(\lambda)\ \frac{1 - V(\lambda)V(\lambda)^-}{1 - \lambda\lambda^-}\ U(\lambda)^-$$

of the space is

$$U(\lambda)\ \frac{1 - V(\lambda)V(\lambda)^-}{1 - \lambda\lambda^-}\ U(\lambda)^-\ \frac{1 - U(\lambda)U(\lambda)^-}{1 - \lambda\lambda^-}\ U(\lambda)\ \frac{1 - V(\lambda)V(\lambda)^-}{1 - \lambda\lambda^-}\ U(\lambda)^-$$
$$+\frac{1 - U(\lambda)U(\lambda)^-}{1 - \lambda\lambda^-}\ U(\lambda)\ \frac{1 - V(\lambda)V(\lambda)^-}{1 - \lambda\lambda^-}\ U(\lambda)^-\ \frac{1 - U(\lambda)U(\lambda)^-}{1 - \lambda\lambda^-},$$

the inequality

$$1 \leq [1 - U(\lambda)U(\lambda)^-]^{-1} + [1 - V(\lambda)V(\lambda)^-]^{-1}$$

is satisfied.

A converse result holds.

Theorem 8. *If $\mathcal{D}(U)$ is the state space of a canonical unitary linear system with transfer function $U(z)$ and if $\mathcal{D}(V)$ is the state space of a canonical unitary linear system with transfer function $V(z)$ such that the inequality*

$$1 \leq [1 - |U(z)|^2]^{-1} + [1 - |V(z)|^2]^{-1}$$

holds when z is in the unit disk, then a Hilbert space $\mathcal{E}$ exists whose elements are the pairs $(u(z), v(z))$ of power series

$$u(z) = f(z) + U(z)h(z)$$

and

$$v(z) = k(z) + V^*(z)g(z)$$

for elements $(f(z), g(z))$ of the space $\mathcal{D}(U)$ and $(h(z), k(z))$ of the space $\mathcal{D}(V)$ and which satisfies the identity

$$\begin{aligned} &\langle (u(z), v(z)), (u(z), v(z)) \rangle_{sD(W)} \\ &= \langle (f(z), g(z)), (f(z), g(z)) \rangle_{\mathcal{D}(U)} \\ &+ \langle (h(z), k(z)), (h(z), k(z)) \rangle_{\mathcal{D}(V)}. \end{aligned}$$

Canonical coisometric linear systems whose state space is a Hilbert space appear in the estimation theory of injective analytic mappings of the unit disk into itself [7]. The proof of the Bieberbach conjecture [5] and the related estimation theory of powers of Riemann mapping functions [6] are treated as applications of the factorization theory of functions which are analytic and bounded by one in the unit disk. Canonical coisometric linear systems whose state space is a Krein space appear in the estimation theory of injective analytic mappings of the unit disk when the image region is not contained in the disk.

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